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PERIODIC OSCILLATIONS OF A COMPOSITE PENDULUM*

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The existence of periodic oscillations of a composite pendulum at noncritical levels of the energy integral (which has no equilibrium positions) is proved using the principle of least action in the Jacobi form. Estimates are obtained of the number of various libratory and rotational periodic oscillations depending on the total energy.

Works dealing with oscillations of the composite pendulum usually consider the equilibrium position stability and, also, the existence and certain properties of periodic and asymptotic motions in the neighborhood of these positions (see, e.g., /1/). The possibility of using methods of the Morse theory for proving the existence of an infinite number of various periodic motions of a composite pendulum for fairly large fixed constant energy was, apparently, first mentioned in /2/.

1. Introduction. The composite pendulum is a mechanical system consisting of n rods connected by hinged joints in a homogeneous gravitational field (Fig.1). No constraints are imposed on the distribution of the mass of rods. The configuration space of the system is the n-dimensional torus $\mathbf{T}^n$, and the angles of rods $q_1, \ldots, q_n$, to the vertical may be taken as the generalized coordinates. The configuration space may be assumed to be Euclidean $\mathbf{R}^n\{q_1, \ldots, q_n\}$, bearing in mind that the points in $\mathbf{R}^n$, whose q-coordinates differ by 2π, correspond to identical positions of the system.

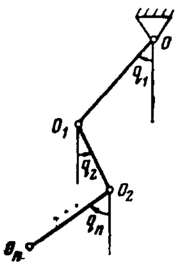

Fig.1

The kinetic energy

$$T(\mathbf{q}, \mathbf{q}^\cdot) = \Sigma a_{ij}(\mathbf{q}) q_i^\cdot q_j^\cdot$$

is a positive definite symmetric form whose coefficients $a_{ij} = a_{ji}$ are periodic in variables $q_1, \ldots, q_n$ of period 2π. The gravitational field potential $V(q_1, \ldots, q_n)$ is also a function periodic in each argument and of period 2π.

The equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{q}^\cdot} - \frac{\partial L}{\partial \mathbf{q}} = 0, \quad L = T + V \tag{1.1}$$

have as their first integral the energy integral $T - V = h$. Since $T \geqslant 0$, hence for fixed total energy h the motion occurs in region $D = \{h + V(\mathbf{q}) \geqslant 0\}$ which is called the region of possible motions. The set $\{h + V(\mathbf{q}) = 0\}$ represents the boundary ∂D of region D.

Only the points $\mathbf{q} = (m_1\pi, \ldots, m_n\pi)$ ($m_1, \ldots, m_n$ are integers) represent critical points of the potential V. They are all isolated and nondegenerate. Critical points correspond to equilibrium positions of the n-link pendulum. We call critical the values of total energy that correspond to these particular solutions. For noncritical values of h the boundary ∂D is a smooth $(n-1)$-dimensional manifold.

The solution of equations of motion at the noncritical level of the energy integral is periodic then and only then when its trajectory on $\mathbf{T}^n$ is closed. As shown in /3/, the trajectories of real systems can be of two types: they either intersect the boundary ∂D or have with it exactly two common points. Solutions of the first type may be reasonably called rotations, and of the second, librations (for details see /3/).

2. Some subsidiary statements. Let us consider a real mechanical system whose configuration space is $\mathbf{R}^n\{q_1, \ldots, q_n\}$; let $T(\mathbf{q}, \mathbf{q}^\cdot)$ be the kinetic energy and V the potential of the force field.

We denote by Λ_a the transformation of $\mathbf{R}^n$ relative to some point $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{R}^n$: $\mathbf{q} \mapsto -\mathbf{q} + 2\mathbf{a}$, and assume the system to be invariant with respect to the Λ_a transform, i.e. $T(\Lambda_a \mathbf{q}, \mathbf{q}^\cdot) = T(\mathbf{q}, \mathbf{q}^\cdot)$ and $V(\Lambda_a \mathbf{q}) = V(\mathbf{q})$.

Lemma 1. If the trajectory of some solution $\mathbf{q}(t)$ passes through point $\mathbf{a}$ ($\mathbf{q}(0) = \mathbf{a}$), then that curve is invariant to transform Λ_a (i.e. $\mathbf{q}(-t) = \Lambda_a \mathbf{q}(t) = -\mathbf{q}(t) + 2\mathbf{a}$)), and $\mathbf{q}^\cdot(-t) = \mathbf{q}^\cdot(t)$.

Proof. Since the kinetic energy T is a quadratic form of generalized velocities and the system is invariant to transform Λ_a, function $\mathbf{q}'(t) = -\mathbf{q}(t) + 2\mathbf{a}$ is also a solution of the equations of motion (1.1). Let us set $\mathbf{q}''(t) = \mathbf{q}(-t)$. Since $\mathbf{q}'(0) = \mathbf{q}''(0)$ and $\mathbf{q}'^\cdot(0) = \mathbf{q}''^\cdot(0)$,

*Prikl.Matem.Mekhan.,44,No.2,238-244,1980 hence on the strength of the theorem on the uniqueness of solutions of Eqs. (1.1) $\mathbf{q}'(t) \equiv \mathbf{q}''(t)$ for all $t \in \mathbf{R}$.

Lemma 2. Let us assume that the considered system is invariant to transform Λ_b ($\mathbf{b} \in \mathbf{R}^n$, $\mathbf{a} \neq \mathbf{b}$). If the trajectory of some solution $\mathbf{q}(t)$ passes through points $\mathbf{a}$ and $\mathbf{b}$, then

1) there exists a $\tau > 0$ such that $\mathbf{q}(t+\tau) = \mathbf{q}(t) + 2(\mathbf{b}-\mathbf{a})$ for all $t \in \mathbf{R}$, and
2) the velocity $\mathbf{q}^{\cdot}(t)$ is never zero.

Proof. 1) Since the system is invariant to transforms Λ_a and Λ_b, it is invariant to their grouping $\Lambda = \Lambda_b\Lambda_a : \mathbf{q} \mapsto \mathbf{q} + 2(\mathbf{b}-\mathbf{a})$. Hence function $\mathbf{q}'(t) = \mathbf{q}(t) + 2(\mathbf{b}-\mathbf{a})$ is a solution of Eqs. (1.1). Let us assume for definiteness that $\mathbf{q}(0) = \mathbf{a}$. According to Lemma 1 function $\mathbf{q}(\tau)$ is at some instant of time τ equal $\mathbf{c} = \Lambda_b\mathbf{a} = \mathbf{a} + 2(\mathbf{b}-\mathbf{a})$ and $\mathbf{q}^{\cdot}(0) = \mathbf{q}^{\cdot}(\tau)$. Since the system is autonomous, function $\mathbf{q}''(t) = \mathbf{q}(t+\tau)$ is also a solution of Eqs. (1.1). Since $\mathbf{q}'(0) = \mathbf{q}''(0)$ and $\mathbf{q}'^{\cdot}(0) = \mathbf{q}''^{\cdot}(0)$, hence $\mathbf{q}'(t) = \mathbf{q}''(t)$ for all $t \in \mathbf{R}$.

2) Let us assume that $\mathbf{q}(t_1) = \mathbf{a}$, $\mathbf{q}(t_2) = \mathbf{b}$ and $t_2 > t_1$. It is obviously sufficient to show that $\mathbf{q}^{\cdot}(t) \neq 0$ in the interval (t_1, t_2). Let us assume the opposite, i.e. that at some instant of time $\tau \in (t_1, t_2)$ the velocity $\mathbf{q}^{\cdot}(\tau) = 0$. Since system (1.1) is reversible, point $\mathbf{q}$ will subsequently move along the same trajectory but in the opposite direction (see, e.g., /1/). In accordance with Lemma 1 solution $\mathbf{q}(t)$ represents a libration whose trajectory does not contain point $\mathbf{b}$. But this contradicts the assumptions of Lemma 2.

According to the principle of least action in the Jacobi form the trajectories of solutions of Eqs. (1.1) inside region D of possible motions are geodesic lines of metric $dp = [h + V(\mathbf{q})]^{1/2}ds$, where ds is the Riemann metric on $\mathbf{R}^n$ which defines the kinetic energy (i.e. $T = (ds/dt)^2/2$). We call the distance $d(\mathbf{a}, \mathbf{b})$ between points $\mathbf{a}, \mathbf{b} \in D$ the lower bound of lengths in metric dp of piecewise smooth curves from region D whose ends are at points $\mathbf{a}$ and $\mathbf{b}$. The quantity

$$\partial(\mathbf{a}) = \inf_{\mathbf{b} \in \partial D} d(\mathbf{a}, \mathbf{b})$$

defines the distance from point $\mathbf{a} \in D$ to the boundary ∂D (for details see /4/). We assume that the considered here system has no equilibrium positions along the boundary ∂D.

Lemma 3. For any point $\mathbf{a}$ of the compact region D Eqs. (1.1) have a solution $\mathbf{q}(t)$ such that $\mathbf{q}(0) = \mathbf{a}$, $\mathbf{q}(\tau) \in \partial D$, and the length of the non-self-intersecting curve $\mathbf{q}(t)$, $t \in [0, \tau]$ is exactly equal $\partial(\mathbf{a})$.

Lemma 4. If $d(\mathbf{a}, \mathbf{b}) < \partial(\mathbf{a}) + \partial(\mathbf{b})$, then Eqs. (1.1) have a solution $\mathbf{q}(t)$ such that $\mathbf{q}(t_1) = \mathbf{a}$ and $\mathbf{q}(t_2) = \mathbf{b}$, and the length of the non-self-intersecting curve $\mathbf{q}(t)$, $t \in [t_1, t_2]$ is exactly equal $d(\mathbf{a}, \mathbf{b})$.

Lemmas 3 and 4 were proved in /4/.

The following statements are corollaries of Lemmas 1 and 3.

Statement 1. Let us assume that the following conditions are satisfied:

1) region $D \in \mathbf{R}^n$ is compact,
2) there are no equilibrium positions along the boundary ∂D, and
3) the system is invariant to transform Λ_a ($\mathbf{a} \in D \setminus \partial D$).

Region D contains a libration whose trajectory passes through point $\mathbf{a}$.

Let us consider, as an example, the problem of periodic oscillations in the nonlinear system defined by the equations /5/

$$m_1x^{\cdot\cdot} = f(x) - F(x-y), \quad m_2y^{\cdot\cdot} = g(y) + F(x-y) \quad (m_1, m_2 > 0) \tag{2.1}$$

where functions f, g, and F are assumed odd.

System (2.1) may be written in the form of Lagrange equations with the Lagrangian

$$L = \tfrac{1}{2}(m_1x^{\cdot 2} + m_2y^{\cdot 2}) + V(x, y), \qquad V = \int f(x)\,dx + \int g(y)\,dy - \int F(x-y)\,d(x-y)$$

Periodic solutions that pass through point $x = y = 0$ and have two common points with the boundary of the region of possible motions were called in /5/ oscillations of the normal type. Their existence has been established only in some particular cases, for instance, when $m_1 = m_2$ and $f \equiv g$.

It can be shown that function V is invariant to transform Λ_0 of the plane $\mathbf{R}^2\{x, y\} : (x, y) \mapsto (-x, -y)$. If the set $\{h + V \geqslant 0\}$ is compact, then according to statement 1 there always exist in the case of noncritical values of total energy h oscillations of the normal type (librations) of system (2.1). It is possible that when region $\{h + V \geqslant 0\}$ is homeomorphic to a disk, there are at least two such oscillations.

3. Periodic oscillations of an n-link pendulum.

Let us set

$$h^- = \min_{\mathbf{T}^n} V, \quad h^+ = \max_{\mathbf{T}^n} V$$

Statement 2. If h is a noncritical value (of energy) in the interval (h^-, h^+), then at least one trajectory of the libration periodic solution passes through every critical point of potential V inside region $\{h + V(\mathbf{q}) \geqslant 0\} \subset \mathbf{T}^n$. Different solutions pass through different critical points.

Corollary. If h is a noncritical value in the interval (h^-, h^+), the number of various librations in region $D = \{h + V \geqslant 0\}$ is not less than the number of critical points inside D.

Since for all $h \geqslant h^-$ point $\mathbf{q} = (0, \ldots, 0) \in D$, hence for $h^- < h < h^+$ the librations of an n-link pendulum always exist. When $h < h^+$ and is fairly close to h^+, there are at least $2^n - 1$ librations in region D.

Remark. These statements enhance the estimates of the number of various librations of an n-link pendulum obtained in /6/.

Proof of Statement 2. Let $\mathbf{a} \in \mathbf{R}^n$ be a critical point of potential V. It is possible to show that the kinetic energy $T(\mathbf{q}, \mathbf{q}^{\cdot})$ and the potential $V(\mathbf{q})$ are invariant to transform Λ_a. According to Lemma 3 it is possible to connect point $\mathbf{a} \in \{h + V \geqslant 0\} \subset \mathbf{T}^n$ to some point of the boundary by a segment of the minimal geodesic γ_a, to which corresponds the geodesic Γ_a of Jacobi's metric in region $\{h + V \geqslant 0\} \subset \mathbf{R}^n$, which connects point $\mathbf{a} \in \mathbf{R}^n$ to some point of ∂D. Let us set $\Gamma_a' = \Lambda_a \Gamma_a$. According to Lemma 1 curve $\Gamma = \Gamma_a \cup \Gamma_a'$ is the trajectory of some libration in region $D \subset \mathbf{R}^n$. The curve γ on torus $\mathbf{T}^n$ which is the sought libration passing through point $\mathbf{a} \in \mathbf{T}^n$ obviously corresponds to curve Γ. The librations that pass through different critical points are different, since by Lemma 2 the velocity of motion would, otherwise, never vanish.

Let us consider the case when $h > h^+$. Since then $\partial D = \varnothing$, periodic motions can only be rotations. Let us investigate the problem of existence of periodic rotations of an n-link pendulum whose k-th link makes N_k complete turns ($N_1, \ldots, N_n$ are fixed integers) during one period. We shall call such motions rotations of type $[N_1, \ldots, N_n]$.

Statement 3. For any fixed integers $N_1, \ldots, N_n$ and any $h > h^+$ there exist 2^{n-1} different periodic rotations of the $[N_1, \ldots, N_n]$ type, with total energy h, whose trajectories on $\mathbf{T}^n$ pass through pairs of critical points of potential V.

Proof. Let us, first, assume that the numbers $N_1, \ldots, N_n$ are coprime, and consider in the space $\mathbf{R}^n$ a pair of critical points $\mathbf{a}'$ and $\mathbf{a}''$ of the potential V, whose q_k-coordinates differ by πN_k. These points considered as points of $\mathbf{T}^n$ different. Let, for example,

$$\mathbf{a}' = (m_1' \pi, \ldots, m_n' \pi), \quad \mathbf{a}'' = (m_1'' \pi, \ldots, m_n'' \pi)$$

If $h > h^+$, the Riemannian space $(\mathbf{R}^n, dp)$ is complete /7/, hence, by the Hopf—Rinov theorem points $\mathbf{a}'$ and $\mathbf{a}''$ can be connected by the shortest geodesic to which corresponds solution $\mathbf{q}(t)$ of Eqs. (1.1) such that $\mathbf{q}(t') = \mathbf{a}'$ and $\mathbf{q}(t'') = \mathbf{a}''$ $(t'' > t')$. Since the problem is invariant to transforms $\Lambda_{a'}$ and $\Lambda_{a''}$, hence by Lemma 2 there exists a number $\tau > 0$ such that

$$\mathbf{q}(t + \tau) - \mathbf{q}(t) = 2(\mathbf{a}'' - \mathbf{a}') = (2N_1\pi, \ldots, 2N_n\pi)$$
$$N_1 = m_1'' - m_1', \ldots, N_n = m_n'' - m_n'$$

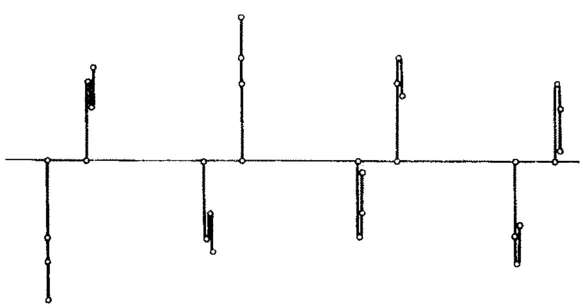

Fig.2

Hence the trajectory of the respective solution on torus $\mathbf{T}^n$ is closed, and that solution is a periodic rotation of the $[N_1, \ldots, N_n]$ type of period τ. Since the numbers $N_1, \ldots, N_n$ are coprime, τ is the least period of solution $\mathbf{q}(t)$ and the remaining periods are its multiples.

Let us assume that the obtained solution $\mathbf{q}(t)$ passes through the critical point $\mathbf{c}$. Then according to Lemma 2 there exist τ' and τ'' such that

$$\mathbf{q}(t + \tau') - \mathbf{q}(t) = 2(\mathbf{c} - \mathbf{a}'), \quad \mathbf{q}(t + \tau'') - \mathbf{q}(t) = 2(\mathbf{c} - \mathbf{a}'')$$

for all $t \in \mathbf{R}$. Hence the periods τ' and τ'' of that solution are multiples of τ, and $\tau' - \tau'' = \tau$. This implies the existence of an integer p such that $\mathbf{c} - \mathbf{a}' = (pN_1\pi, \ldots, pN_n\pi)$. Then $\mathbf{c} - \mathbf{a}'' = ((p-1)N_1\pi, \ldots, (p-1)N_n\pi)$, which means that point $\mathbf{c}$ considered as a point of the n-dimensional torus $\mathbf{T}^n$ coincides with one of points $\mathbf{a}', \mathbf{a}'' \in \mathbf{T}^n$.

Thus all critical points of potential V on $\mathbf{T}^n$ split into pairs through which pass the trajectories of periodic rotations of type $\{N_1, \ldots, N_n\}$, and none of these trajectories contain points of other pairs. Since the over-all number of critical points is 2^n, the number of different rotations of the pendulum that belong to the considered type is 2^{n-1}.

Let us assume now that the numbers $N_1, \ldots, N_n$ are not coprime and that $p > 1$ is their greatest common divisor. We set $N_1 = pN_1', \ldots, N_n = pN_n'$. According to the just proved there exists 2^{n-1} different rotations of type $\{N_1', \ldots, N_n'\}$. Consider the periodic rotations derived from solutions of type $\{N_1', \ldots, N_n'\}$ by an p-tuple increase of the period. They are all different and, evidently, of type $\{N_1, \ldots, N_n\}$. The statement is proved.

Four pairs of equilibrium positions of a three-link pendulum are shown in Fig.2 for various periodic rotations of type /1,2,3/.

We shall show in conclusion that under certain conditions periodic rotations of a pendulum also exist when $h < h^+$. To prove this let us consider a two-link pendulum whose rods are of equal length l and their mass concentrated at points O_1 and O_2 equal, respectively, m_1 and m_2. It can be shown that

$$T = \tfrac{1}{2}(m_1 + m_2) l^2 \dot{q}_1^2 + \tfrac{1}{2} m_2 l^2 \dot{q}_2^2 + m_2 l^2 \dot{q}_1 \dot{q}_2 \cos(q_1 - q_2)$$
$$V = m_1 g l \cos q_1 + m_2 g l (\cos q_1 + \cos q_2)$$

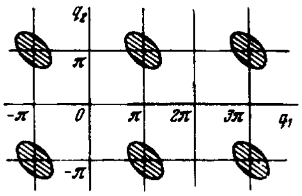

Fig.3

Let us consider the case when the quantity h is fairly close to h^+. The region of possible motions in the plane $\mathbf{R}^2\{q_1, q_2\}$ is shown in Fig.3 (unshaded). We fix the value of m_1 and make mass m_2 approach zero. When m_2 is fairly small, the distance between points $\mathbf{a} = (0, 0)$ and $\mathbf{b} = (0, \pi)$ is less than the sum of distances from these points to the boundary of the region of possible motions.

Actually, $d(\mathbf{a}, \mathbf{b})$ does not exceed the length of segment $\{q_1 = 0, q_2 \in [0, \pi]\} \subset R^2$ which is equal

$$\sqrt{m_2}\, l \int_0^\pi [h + m_1 g l + m_2 g l (1 + \cos q_2)]^{1/2} dq_2$$

and approaches zero as $m_2 \to 0$. For small m_2 the region of possible motions differs from $\{h + m_1 g l \cos q_1 \geqslant 0\}$. Consequently,

$$\lim_{m_2 \to 0} \partial(\mathbf{a}) = \frac{\sqrt{m_1}\, l}{2} \oint [h + m_1 g l \cos q_1]^{1/2} dq_1 > 0$$

which means that for small m_2 the inequality $d(\mathbf{a}, \mathbf{b}) < \partial(\mathbf{a}) + \partial(\mathbf{b})$ is satisfied.

Lemma 4 implies the existence of the shortest geodesic metric $d\rho$ that links points a and b lying inside the region of possible motions. To this geodesic corresponds the solution of Eqs. (1.1) at the level of the energy integral with constant h. Since the system is invariant to transforms Λ_a and Λ_b, hence in accordance with Lemma 2 the obtained solution represents periodic rotation.

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